数理解析研究所講究録
第 1836 巻 2013 年 1-6

EXTENDING GEODESICS IN THE CURVE COMPLEX

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1. Introduction

The *curve complex* (or *complex of curves*) is introduced by Harvey [3] who was studying the Teichmüller space of Riemann surfaces. The *distance* between curves on a surface can be defined by using the curve complex of the surface. Masur and Minsky, in [6, 7, 8] for example, studied various properties of the curve complexes and used them to investigate 3-manifolds. They introduced the notion of subsurface projection map which is one of the useful tools to treat the curve complexes. From the view point of the curve complexes, Hempel [4] defined so-called *distance* as a measure of complexities of Heegaard splittings of 3-manifolds.

Since the curve complex has complicated structure in general (it is not even locally finite), it is not easy to estimate the distance between given two curves. It is even more difficult in general to estimate the distance of a Heegaard splitting since we need to estimate the distance between two sets of infinitely many vertices in the curve complex. Although, it is known that the curve complexes have infinite diameter (see [6]) and that there exist Heegaard splittings of arbitrarily high distance (see [1, 2, 4] for example). Namely, for any given integer n, there exists a Heegaard splitting with distance bigger than n.

In this paper, we give a method to extend a geodesic to one with given length (see Section 3) and use it to construct Heegaard splittings with distance n for any given non-negative integer n (see Section 4).

2. Definitions and notations

2.1. **Curve complexes.** Let S be a compact orientable surface with genus g and p boundary components. A simple closed curve in S is *essential* if it does not bound a disk in S and is not parallel to a component of ∂S. An arc properly embedded in S is *essential* if it does not co-bound a disk in S together with an arc on ∂S. We say that S is *sporadic* if $g=0, p\leq 4$ or $g=1, p\leq 1$.

Except in sporadic cases, the *curve complex* $\mathcal{C}(S)$ is defined as follows: each vertex of $\mathcal{C}(S)$ is the isotopy class of an essential simple closed curve on S, and a collection of $k+1$ vertices forms a k-simplex of $\mathcal{C}(S)$ if they can be realized by disjoint curves in S. In sporadic cases, we need to modify the definition of the curve complex slightly, as follows. We assume that S is a torus, a torus with one boundary component, or a sphere with 4 boundary components since, otherwise, there are no essential simple closed curves in S. When S is a torus or a torus with one boundary component (resp. a sphere with 4 boundary components), a collection of $k+1$ vertices forms a k-simplex of $\mathcal{C}(S)$ if they can be realized by curves in S which mutually intersect exactly once (resp. twice). The *arc-and-curve complex* $\mathcal{AC}(S)$ is defined similarly, as follows: each vertex of $\mathcal{AC}(S)$ is the

Received December 31, 2012.

isotopy class of an essential properly embedded arc or an essential simple closed curve on S, and a collection of $k+1$ vertices forms a k-simplex of $\mathcal{AC}(S)$ if they can be realized by disjoint arcs or simple closed curves in S.

We can define the *distance* between two vertices in the curve complex $\mathcal{C}(S)$ to be the minimal number of 1-simplexes of a simplicial path in $\mathcal{C}(S)$ joining the two vertices. We denote by $d_{\mathcal{C}(S)}(x,y)$, or $d_S(x,y)$ in brief, the distance in $\mathcal{C}(S)$ between the vertices x and y. For subsets X and Y of the vertices of $\mathcal{C}(S)$, we define $\mathrm{diam}_S(X,Y) = \mathrm{diam}_S(X \cup Y)$. Similarly, we can define the distance $d_{\mathcal{AC}(S)}(x,y)$ and $\mathrm{diam}_{\mathcal{AC}(S)}(X,Y)$. We denote by $[a_0, a_1 \ldots, a_n]$ the path in $\mathcal{C}(S)$ with vertices $a_0, a_1 \ldots, a_n$ such that $a_i \cap a_{i+1} = \emptyset$ $(i = 0, 1, \ldots, n-1)$. We call a path $[a_0, a_1 \ldots, a_n]$ a *geodesic* if $n = d_s(a_0, a_n)$.

2.2. **Subsurface projections.** Let $\mathcal{P}(Y)$ denote the power set of a set Y. Suppose that X is an essential subsurface of S that contains an essential simple closed curve. We call the composition $\pi_0 \circ \pi_A$ of maps $\pi_A : \mathcal{C}^0(S) \to \mathcal{P}(\mathcal{AC}^0(X))$ and $\pi_0 : \mathcal{P}(\mathcal{AC}^0(X)) \to \mathcal{P}(\mathcal{C}^0(X))$ a *subsurface projection* if they satisfy the following (see Figure 1): for a vertex α, take a representative α so that $|\alpha \cap X|$ is minimal, where $|\cdot|$ is the number of connected components. Then

- $\pi_A(\alpha)$ is the set of all isotopy classes of the components of $\alpha \cap X$,
- $\pi_0(\{\alpha_1, \ldots, \alpha_n\})$ is the union for all $i = 1, \ldots, n$ of the set of all isotopy classes of the components of $\partial N(\alpha_i \cup \partial X)$ which are essential in X, where $N(\alpha_i \cup \partial X)$ is a regular neighborhood of $\alpha_i \cup \partial X$ in X.

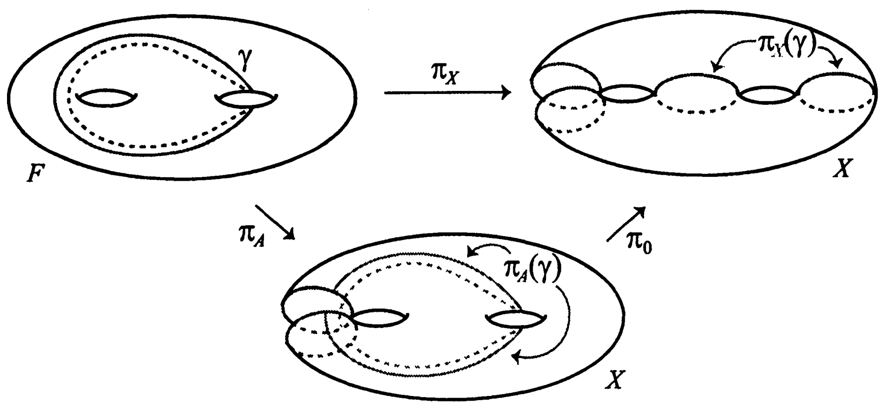

FIGURE 1

The following lemma can be proved by using [7, Lemma 2.2].

Lemma 2.1 (cf. [5, Lemma 2.1]). *Let X be a subsurface of a surface S as above. Let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a path in $\mathcal{C}(S)$ such that $\alpha_i \cap X \neq \emptyset$ for each $i = 0, 1, \ldots, n$. Then* $\mathrm{diam}_X(\pi_X(\alpha_0), \pi_X(\alpha_n)) \leq 2n$.

Remark 2.2. In the above lemma, the assumption that $\alpha_i \cap X \neq \emptyset$ for each $i = 0, 1, \ldots, n$ is necessary. For example, consider a path $[\alpha_0, \alpha_1, \alpha_2]$ in $\mathcal{C}(S)$ such that $\alpha_0, \alpha_2 \subset X$ and $\alpha_1 \subset S \setminus X$. Since we can choose α_0 and α_2 so that $d_X(\alpha_0, \alpha_2)(= \mathrm{diam}_X(\pi_X(\alpha_0), \pi_X(\alpha_2)))$ is arbitrarily high, the assertion of the above lemma does not hold in this case.

3. Extending geodesics

Let S be a compact orientable surface. Let α_0, α_1 and α_2 be simple closed curves on S such that $\alpha_0 \cap \alpha_1 = \emptyset$, $\alpha_1 \cap \alpha_2 = \emptyset$ and $\alpha_0 \cap \alpha_2 \neq \emptyset$. Then $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic in $\mathcal{C}(S)$. In this section, we show how to extend this geodesic to one with length n for a given integer $n(> 2)$. To this end, we further assume that, for $i = 0, 2$, either α_i is non-separating on S or α_i cuts S into two surfaces one of which is a 3-holed sphere. (We need this assumption to use Lemma 2.1.)

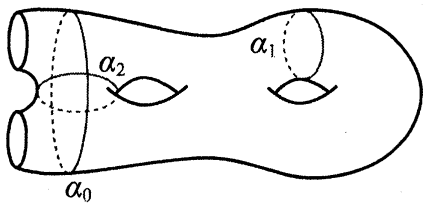

FIGURE 2

Let X_2 be the complement of an open neighborhood of α_2 in S. Note that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic of length two in $\mathcal{C}(S)$. Choose a homeomorphism $f_2 : S \to S$ such that $f_2(\alpha_2) = \alpha_2$ and that $\mathrm{diam}_{X_2}(\pi_{X_2}(\alpha_0), \pi_{X_2}(f_2(\alpha_0))) > 8$. (This is possible by [6, Proposition 4.6], for example.) Let $\alpha_3 = f_2(\alpha_1)$ and $\alpha_4 = f_2(\alpha_0)$. Note that $[\alpha_2, \alpha_3, \alpha_4]$ is a geodesic of length two in $\mathcal{C}(S)$.

Claim 3.1. *The path $[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4]$ constructed above is a geodesic in $\mathcal{C}(S)$.*

Proof. Let $[\beta_0, \beta_1, \beta_2, \ldots, \beta_m]$ be a geodesic in $\mathcal{C}(S)$ such that $\beta_0 = \alpha_0$, $\beta_m = \alpha_4$. Then $m \leq 4$.

Assume that $\beta_j \neq \alpha_2$ for any $j = 0, 1, \ldots, m$. Then $\beta_j \cap X_2 \neq \emptyset$ for each $j = 0, 1, \ldots, m$. By Lemma 2.1, we have $\mathrm{diam}_{X_2}(\pi_{X_2}(\beta_0), \pi_{X_2}(\beta_m)) \leq 2m \leq 8$, a contradiction. Hence, $\beta_j = \alpha_2$ for some $j = 0, 1, \ldots, m$.

We have the equalities

$$\begin{aligned} j = d_S(\beta_0, \beta_j) &= d_S(\alpha_0, \alpha_2) = 2, \\ m - j = d_S(\beta_j, \beta_m) &= d_S(\alpha_2, \alpha_4) = 2. \end{aligned}$$

By combining the above equalities, we have $m = 4$, and hence, $[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4]$ is a geodesic in $\mathcal{C}(S)$. □

Remark 3.2. The above claim is an easy generalization of the example given in [9, Chapter 2, Section 6]. In the example, a geodesic in $\mathcal{C}(S)$ of length 4 is constructed when S is a 5-holed sphere.

We repeat this procedure to construct a geodesic $[\alpha_0, \alpha_1, \alpha_2 \ldots, \alpha_n]$ for any even integer n. Namely, for each $i \in \{2, 4, \ldots, n-2\}$, assume that $[\alpha_0, \alpha_1, \alpha_2 \ldots, \alpha_i]$ is a geodesic in $\mathcal{C}(S)$ and that either α_i is non-separating on S or α_i cuts S into two surfaces one of which is a 3-holed sphere.

- X_i is the complement of an open neighborhood of α_i in S,
- $f_i : S \to S$ is a homeomorphism such that $f_i(\alpha_i) = \alpha_i$ and that

$$\mathrm{diam}_{X_i}(\pi_{X_i}(\alpha_{i-2}), \pi_{X_i}(f_i(\alpha_{i-2}))) > 2(i+2),$$

- $\alpha_{i+1} = f_i(\alpha_{i-1})$ and $\alpha_{i+2} = f_i(\alpha_{i-2})$.

Note that $[\alpha_i, \alpha_{i+1}, \alpha_{i+2}]$ is a geodesic of length two in $\mathcal{C}(S)$.

Claim 3.3. *The path $[\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{i+2}]$ constructed above is a geodesic in $\mathcal{C}(S)$.*

Proof. Let $[\beta_0, \beta_1, \beta_2, \dots, \beta_m]$ be a geodesic in $\mathcal{C}(S)$ such that $\beta_0 = \alpha_0$, $\beta_m = \alpha_{i+2}$. Then $m \leq i+2$.

Assume that $\beta_j \neq \alpha_i$ for any $j = 0, 1, \dots, m$. Then $\beta_j \cap X_i \neq \emptyset$ for each $j = 0, 1, \dots, m$. By Lemma 2.1, we have $\mathrm{diam}_{X_i}(\pi_{X_i}(\beta_0), \pi_{X_i}(\beta_m)) \leq 2m \leq 2(i+2)$, a contradiction. Hence, $\beta_j = \alpha_i$ for some $j = 0, 1, \dots, m$.

We have the equalities

$$\begin{aligned} j = d_S(\beta_0, \beta_j) &= d_S(\alpha_0, \alpha_i) = i, \\ m - j = d_S(\beta_j, \beta_m) &= d_S(\alpha_i, \alpha_{i+2}) = 2. \end{aligned}$$

By combining the above equalities, we have $m = i + 2$, and hence, $[\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{i+2}]$ is a geodesic in $\mathcal{C}(S)$. □

In this way, we can extend the geodesic $[\alpha_0, \alpha_1, \alpha_2]$ to one with any given length.

4. Application to Heegaard splittings

A 3-manifold V is a *compression body* if there exists a closed (possibly empty) surface F and a 0-handle B such that V is obtained from $F \times [0,1] \cup B$ by adding 1-handles to $F \times \{1\} \cup \partial B$. The subsurface of ∂V corresponding to $F \times \{0\}$ is denoted by $\partial_- V$, and we denote by $\partial_+ V$ the subsurface $\partial V \setminus \partial_- V$ of ∂V. For a compact orientable 3-manifold M, we say that $C_1 \cup_P C_2$ is a *genus-g Heegaard splitting* of M if C_1 and C_2 are compression bodies in M such that $C_1 \cup C_2 = M$ and $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = P$.

For a compression body V, the *disk complex* $\mathcal{D}(V)$ is the subcomplex of $\mathcal{C}(\partial_+ V)$ consisting of the vertices with representatives bounding disks of V. For a genus-$g(\geq 2)$ Heegaard splitting $C_1 \cup_P C_2$, the (Hempel) *distance* of $C_1 \cup_P C_2$ is defined to be $d_P(\mathcal{D}(C_1), \mathcal{D}(C_2)) = \min\{d_P(x, y) \mid x \in \mathcal{D}(C_1), y \in \mathcal{D}(C_2)\}$.

Since Hempel [4] introduced the notion of distance of Heegaard splittings, the existence of Heegaard splittings with arbitrarily high distance has been shown by using various methods (see [4, 1, 2] for example). We can use a geodesic constructed in the previous section to prove the following.

Theorem 4.1 ([5, Theorem 1.1]). *For any integer $n > 0$ and any integer $g > 1$, there exists a genus-g Heegaard splitting $C_1 \cup_P C_2$ with distance exactly n.*

We remark that compression bodies C_1 and C_2 constructed in the proof of the above theorem have boundary components other than the Heegaard surface $P = \partial_+ C_1 = \partial_+ C_2$. The following proposition is also useful to prove the above theorem.

Proposition 4.2 ([5, Proposition 3.1]). *Let V be a compression body obtained by adding a 1-handle to $F \times [0,1]$, where F is a genus-$(g-1)$ closed orientable surface ($g > 1$). Then we have the following.*

1. *There is a unique non-separating disk in V up to ambient isotopy.*
2. *Any essential separating disk in V can be isotoped to be disjoint from the non-separating disk in V.*

In the remaining of this paper, we give a brief explanation on how to construct the Heegaard splitting in the above theorem. We show the proof only in case where n is even. In case where n is odd, we need to modify slightly the way to extend a geodesic in the curve complex (see [5, Subsection 4.2]).

Let C_1 and C_2 be copies of the compression body obtained by adding a 1-handle to $F \times [0,1]$, where F is a genus-$(g-1)$ closed orientable surface $(g > 1)$. We construct a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_{n+2}]$ of length $(n+2)$ in $\mathcal{C}(\partial_+ C_1)$ as in the previous section. We may assume that α_0 is the boundary of the non-separating essential disk D_1 properly embedded in C_1, and we further assume that α_2 is a simple closed curve on $\partial_+ C_1$ which intersects α_0 transversely in one point. This implies that α_{n+2} intersects α_n transversely in one point by the construction. Take any homeomorphism $h : \partial_+ C_2 \to \partial_+ C_1$ such that $h(\partial D_2) = \alpha_{n+2}$, where D_2 is the non-separating essential disk properly embedded in C_2. We identify the boundary components $\partial_+ C_1$ and $\partial_+ C_2$ by h, and let $P = \partial_+ C_1 = h(\partial_+ C_2)$. Then $C_1 \cup_P C_2$ is a genus-g Heegaard splitting of a compact orientable 3-manifold.

By Proposition 4.2, the boundary of any essential disk in C_1 (resp. C_2) has distance at most 1 from α_0 (resp. α_{n+2}). Thus we can see that the distance of the Heegaard splitting $C_1 \cup_P C_2$ is at least n. Note, on the other hand, that α_2 (resp. α_n) intersects α_0 (resp. α_{n+2}) transversely in one point. Then, as illustrated in Figure 3, we can find an essential separating disk in C_1 (resp. C_2) whose boundary is disjoint from $\alpha_0 \cup \alpha_2$ (resp. $\alpha_n \cup \alpha_{n+2}$), which implies that the distance of the Heegaard splitting $C_1 \cup_P C_2$ is at most n.

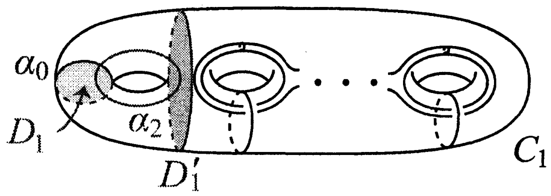

FIGURE 3

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